Problem Bank for Linear Algebra (Jan. 2024)

Textbook: Linear Algebra written by Friedberg, Insel and Spence

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Problem 1. Let V and W be vector spaces over a field F. Let $Z = \{(v, w) \,|\, v \in V \text{ and } w \in W\}$. Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

Problem 2. Let $V = \{(a_1, a_2) \,|\, a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinate-wise, and for (a_1, a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Problem 3. In each part, determine whether the given set is a subspace of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answer.
(1) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \,|\, a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(2) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \,|\, a_1 = a_3 + 2\}$

Problem 4. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \emptyset$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

Problem 5. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Problem 6. In each part, determine whether the given vector is in the span of S.
(1) $(-1, 2, 1)$ and $S = \{(1, 0, 2), (-1, 1, 1)\}$
(2) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ and $S = \{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$

Problem 7. Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$.

Problem 8. Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.
(1) Prove that if $F = \mathbb{R}$, then S is linearly independent.
(2) Prove that if $F = \mathbb{Z}_2$, then S is linearly dependent.

Problem 9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Problem 10. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$.
(1) Prove that if S_1 is linearly dependent, then S_2 is linearly dependent.
(2) Prove that if S_2 is linearly independent, then S_1 is linearly independent.

Problem 11. The vectors $u_1 = (1, 1, 1, 1)$, $u_2 = (0, 1, 1, 1)$, $u_3 = (0, 0, 1, 1)$, and $u_4 = (0, 0, 0, 1)$ form a basis for F^4. Find a unique representation of an arbitrary vector (a_1, a_2, a_3, a_4) in F^4 as a linear combination of u_1, u_2, u_3, and u_4.

Problem 12. Let F be a field and let $W = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in F\}$. Then the set W is a subspace of $M_{2\times 2}(F)$. Find a basis for W.

Problem 13. Find a basis for the subspace $W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}$ of F^5.

Problem 14. Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a is a nonzero scalar, then $\{u + v, au\}$ is also a basis for V.

Problem 15. Let V be a vector space having finite dimension n, and let S be a subset of V with n vectors. Show that S is linearly independent if and only if S spans V.

Problem 16. Justify your answer.
(1) Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$, and $3x - 2$ generate $P_3(\mathbb{R})$?
(2) Is $\{(1, 4, -6), (1, 5, 8), (2, 1, 1), (0, 1, 0)\}$ a linearly independent subset of $\mathbb{R}^3$?

Problem 17. Show that a maximal linearly independent subset of a vector space V is a basis for V.

Problem 18. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a basis for a vector space V over the field $\mathbb{Z}_2$. How many vectors are there in V? Justify your answer.

Problem 19. Give three different bases for $M_{2\times 2}(F)$.

Problem 20. The set of all symmetric matrices in $M_{2\times 2}(\mathbb{R})$ is a subspace of $M_{2\times 2}(\mathbb{R})$. Find the dimension of this subspace.

Problem 21. For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace W of $P_3(\mathbb{R})$ (the vector space of polynomials of degree at most 3 with coefficients in $\mathbb{R}$), where $W = \{f \in P_3(\mathbb{R}) : f(a) = 0\}$.

Problem 22. Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Show that $W_1 \subseteq W_2$ if and only if $\dim(W_1 \cap W_2) = \dim(W_1)$.

Problem 23. Let V and W be vector spaces over a field F, and let $T : V \to W$ be a function.

(1) Prove that T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and for all $c \in F$.

(2) Prove that T is linear if and only if $T(a_1x_1 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$ for all $x_1, \ldots, x_n \in V$ and for all $a_1, \ldots, a_n \in F$.

Problem 24. In the following, prove that T is a linear transformation, and find bases for $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem.
(1) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

(2) $T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$.

(3) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

Problem 25. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Find a basis for $N(T)$.

Problem 26. Let V and W be vector spaces over a field F, let $T : V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

Problem 27. Let V and W be vector spaces over a field F, let $T : V \to W$ be linear. Suppose that T is one-to-one and onto. Prove that $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V if and only if $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Problem 28. Let B be an invertible matrix in $M_{n\times n}(F)$. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$ for every $A \in M_{n\times n}(F)$. Prove that Φ is an isomorphism.

Problem 29. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

(1) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(2) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Problem 30. (1) Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$.

(2) Is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, -1) = (1, 0, 2)$, $T(2, 2) = (1, -1, 4)$ and $T(8, 12) = (3, -5, 12)$? Justify your answer.

(3) Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, 6) = (2, 1)$? Justify your answer.

Problem 31. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations, respectively, defined by

$$T(f(x)) = (3 + x)f'(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(1) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(2) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$ directly. Then use $[U]_\beta^\gamma$ from (1) and Theorem 2.14 to verify your result.

Problem 32. Let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(1) Prove that if UT is one-to-one, then T is one-to-one.

(2) Prove that if UT is onto, then U is onto.

Problem 33. Determine whether the following linear transformation T is invertible? If T is invertible, find T^{-1}.

(1) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.

(2) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Problem 34. (1) Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove that A is invertible and that $B = A^{-1}$. (Hint: Use $L_{AB} = L_A L_B$ and Dimension Theorem.)

(2) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

Problem 35. Let $V = \{\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}\}$. Construct an isomorphism from V to $\mathbb{R}^3$.

Problem 36. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Problem 37. The mapping $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(M) = M^t$ for each $M \in M_{2\times 2}(\mathbb{R})$ is a linear transformation. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, which is the standard ordered basis for $M_{2\times 2}(\mathbb{R})$.

(1) Compute $[T]_\beta$.

(2) Put $A = [T]_\beta$. Verify that $L_A \phi_\beta(M) = \phi_\beta T(M)$ for $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Problem 38. For each of the following matrices, compute the rank and the inverse if it exists.

(1) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$ (2) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$ (3) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

Problem 39. Express the invertible matrix $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ as a product of elementary matrices.

Problem 40. Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then $\operatorname{rank}(cA) = \operatorname{rank}(A)$.

Problem 41. Use A^{-1} to solve the system:
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 4 \end{aligned}$$

Problem 42. Determine whether the system has a solution:
$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 2 \\ x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 &= 4 \end{aligned}$$

Problem 43. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a, b, c) = (a+b, b-2c, a+2c)$. For $v = (1, 3, -2)$, determine whether $v \in R(T)$.

Problem 44. Use Gaussian elimination to solve the system:
$$\begin{aligned} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 &= 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 &= 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 &= 6 \end{aligned}$$

Problem 45. It can be shown that the vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

Problem 46. Let the reduced row echelon form of A be $\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$. Determine A if the first, second, and fourth columns of A are $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, resp.

Problem 47. Find the value of k satisfying the following equation:
$$\det \begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1 + 5c_1 & 3b_2 + 5c_2 & 3b_3 + 5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Problem 48. Evaluate the determinant of the following matrix using elementary row operations.

(1) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

Problem 49. Prove that $\det(kA) = k^n \det(A)$ for any $A \in \mathbf{M}_{n\times n}(F)$.

Problem 50. Use determinants to prove that if $A, B \in \mathbf{M}_{n\times n}(F)$ are such that $AB = I_n$, then A is invertible.

Problem 51. Suppose that $M \in \mathbf{M}_{n\times n}(F)$ can be written in the form $M = \begin{pmatrix} A & B \\ O & I \end{pmatrix}$, where A is a square matrix. Prove that $\det(M) = \det(A)$.

Problem 52. Suppose that $M \in \mathbf{M}_{n\times n}(F)$ can be written in the form $M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$, where A and C are square matrices. Prove that $\det(M) = \det(A)\det(C)$.

Problem 53. Use Cramer's rule to solve the system of linear equations

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= 10 \\ 3x_1 + 4x_2 - 2x_3 &= 0 \end{aligned}$$

Problem 54. Let $T : V \to V$ be a linear operator, where V is an n-dimensional vector space over F. Also let β and γ be two arbitrary ordered bases for V. Prove that $\det([T]_\beta - tI_n) = \det([T]_\gamma - tI_n)$.

Problem 55. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V.

(1) Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

(2) Prove that v is an eigenvector of T corresponding to λ if and only if $[v]_\beta$ is an eigenvector of $[T]_\beta$ corresponding to λ.

Problem 56. For each of the following matrices $A \in \mathbf{M}_{n\times n}(\mathbb{R})$, diagonalize A, if possible.

(1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (2) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ (3) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$

Problem 57. For each linear operator T on V, find an ordered basis γ for V such that $[T]_\beta$ is a diagonal matrix.

(1) $V = \mathbb{R}^3$ and $T(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$

(2) $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$

(3) $V = M_{2\times 2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.

$\circledast$ Let V be an n-dimensional vector space over F and let $T : V \to V$ be a linear operator.

Problem 58. Let W_1 and W_2 be subspaces of V. Prove that the following are equivalent.
(i) $V = W_1 \oplus W_2$
(ii) If γ_1 and γ_2 are bases for W_1 and W_2, respectively, such that $\gamma_1 \cap \gamma_2 = \emptyset$, then the union $\gamma_1 \cup \gamma_2$ is a basis for V.

Problem 59. If λ_1 and λ_2 are distinct eigenvalues of T, prove that $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

Problem 60. Let $A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \in \mathbf{M}_{3\times 3}(\mathbb{R})$. Using diagonalization of A, find an expression for A^n, where n is an arbitrary positive integer.

Problem 61. Prove that if T is diagonalizable and T is invertible, then T^{-1} is also diagonalizable. (Hint: Use that λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.)

Problem 62. Suppose that $A \in \mathbf{M}_{n\times n}(F)$ has three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and that $\dim(E_{\lambda_1}) = n-2$. Prove that A is diagonalizable.

Problem 63. Suppose that $A \in \mathbf{M}_{n\times n}(F)$ is similar to an upper triangular matrix, and that A has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively. Prove that $\det(A) = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}$.

Problem 64. Let $A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix} \in \mathbf{M}_{k\times k}(\mathbb{F})$. Prove that the characteristic polynomial of A is $(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$.

Problem 65. Determine whether the given subspace W is T-invariant or not.

(1) $V = P_3(\mathbb{R})$, $T(f(x)) = xf'(x)$, and $W = P_2(\mathbb{R})$.

(2) $V = \mathbf{M}_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $W = \{A \in V \mid A^t = A\}$.

Problem 66. Let $W_1, W_2, \ldots, W_k$ be T-invariant subspaces of V such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

Prove that T is diagonalizable if T_{W_i} is diagonalizable for all i.

Problem 67. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

(1) $V = \mathbb{R}^4$, $T(a,b,c,d) = (a+b, b-c, a+c, a+d)$, and $z = (1,0,0,0)$.

(2) $V = P_3(\mathbb{R})$, $T(f(x)) = f''(x)$, and $z = x^3$.

Problem 68. For each T-cyclic space W in # 4, compute the characteristic polynomial of T_W in two ways.

Problem 69. Let λ be an eigenvalue of T. Using the definition of generalized eigenspace, prove that K_λ is a subspace of V.

Problem 70. Let λ and μ be distinct eigenvalues of T. Prove that $K_\lambda \cap K_\mu = \emptyset$.

Problem 71. Let $U : V \to V$ be linear.

(1) Prove that $N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$.

(2) Prove that if $N(U^m) = N(U^{m+1})$ for some positive integer m, then $N(U^m) = N(U^k)$ for any positive integer $k \geq m$.

Problem 72. Suppose that $\mathrm{char}(T)$ splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Prove that T is diagonalizable if and only if $N(T - \lambda_i I_V) = N((T - \lambda_i I_V)^2)$ for every $1 \leq i \leq k$.

Problem 73. Let $f(t) = \mathrm{char}(T)$ and $p(t) = \min(T)$. Prove that if $f(t)$ splits, then $f(t)$ divides $[p(t)]^n$, where $n = \dim(V)$.

Problem 74. Find the minimal polynomial of the following matrices.

(1) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (3) $\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{pmatrix}$

Problem 75. Describe all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is diagonalizable and $T^3 - 2T^2 + T = 0$.

Problem 76. Prove Theorem 7.13: Let β be an ordered basis for V. Then $\min(T) = \min([T]_\beta)$.

Problem 77. Let W be a T-invariant subspace of V. Prove that $\min(T_W)$ divides $\min(T)$.

Problem 78. Provide reasons why each of the following is not an inner product on the given vector spaces.

(1) $\langle (a,b), (c,d) \rangle = ac - bd$ on $\mathbb{R}^2$

(2) $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $\mathbf{M}_{2\times 2}(\mathbb{R})$

(3) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t)\,dt$ on $P(\mathbb{R})$, where $'$ denotes differentiation.

Problem 79. Let V be an inner product space. Define a function $\mathrm{d} : V \times V \to \mathbb{R}$ by $\mathrm{d}(x,y) = \|x - y\|$ for every $x, y \in V$. Prove the following statements for all $x, y, z \in V$ (i.e., prove that d is a **distance** on V).

(1) $\mathrm{d}(x,y) \geq 0$.
(2) $\mathrm{d}(x,y) = 0$ if and only if $x = y$.
(3) $\mathrm{d}(x,y) = \mathrm{d}(y,x)$.
(4) $\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + \mathrm{d}(z,y)$.

Problem 80. Give your reason why $\langle A, B \rangle = \mathrm{tr}(AB)$ cannot be an inner product on $\mathbf{M}_{2\times 2}(\mathbb{R})$.

Problem 81. If $\{u, v, w\}$ is an orthonormal set of vectors in an inner product space V over $\mathbb{C}$, compute $\|iu + v + (2+i)w\|$.

Problem 82. Let $T : V \to V$, where V is an inner product space. Prove that if $\|T(x)\| = \|x\|$ for all x, then T is one-to-one.

Problem 83. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$. (Hint: Use Theorem 6.6)

Problem 84. Let W be a finite dimensional subspace of an inner product space V, and let $z \in V$. Suppose that $\{v_1, \ldots, v_k\}$ be an ordered basis for W. If $\langle z, v_i \rangle = 0$ for every $i = 1, \ldots, k$, prove that $z \in W^\perp$.

Problem 85. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$. (Hint: Use Theorem 6.6)

Problem 86. Let W be a finite dimensional subspace of an inner product space V, and let $z \in V$. Suppose that $\{v_1, \ldots, v_k\}$ be an ordered basis for W. If $\langle z, v_i \rangle = 0$ for every $i = 1, \ldots, k$, prove that $z \in W^\perp$.

Problem 87. Let $W = \text{span}(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. Find an orthonormal basis for $W^\perp$.

Problem 88. For $u = (2, 6) \in \mathbb{R}^2$ and a subspace $W = \{(x, y) \mid y = 4x\}$ of $\mathbb{R}^2$, find the orthogonal projection of u on W and the shortest distance from u to W.

Problem 89. Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be a linear transformation defined by $T(f) = f' + 3f$. If $P_1(\mathbb{R})$ is an inner product space with $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$, then evaluate T^* at $y = 4 - 2t$. (Hint: See the proofs of Theorem 6.8 and Theorem 6.9.)

Problem 90. For given set $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$ of data, use the least squares approximation to find the best fits with both (i) a linear function and (ii) a quadratic function.

Problem 91. Let $A \in \mathbf{M}_{m\times n}(F)$. If $W = R(L_{A^*})$, prove that $W^\perp = N(L_A)$.

Problem 92. Let V be a finite-dimensional inner product space. Let W be a subspace of V, and let $\gamma = \{w_1, w_2, \ldots, w_k\}$ be an orthonormal basis for W. Define a linear transformation $T : V \to V$ by $T(v) = \langle v, w_k \rangle w_1 + \langle v, w_1 \rangle w_2 + \langle v, w_2 \rangle w_3 + \cdots + \langle v, w_{k-1} \rangle w_k$ for every $v \in V$. Prove that T is normal.

Problem 93. Let $V = \mathbb{C}^2$ be a vector space over $\mathbb{C}$. Show that V has an orthonormal basis consisting of eigenvectors of $T(a, b) = (2a + (3 + 3i)b, (3 - 3i)a + 5b)$.

Problem 94. Prove that every eigenvalue of $A = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1-i \\ 0 & 1+i & -1 \end{pmatrix}$ is real without using $\text{char}(A)$.

Problem 95. Prove that the composite of unitary [orthogonal, resp.] operators is unitary [orthogonal, resp.].

Problem 96. For $z \in \mathbb{C}$, define $T_z : \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self-adjoint, or unitary.

Problem 97. Let V be the inner product space of complex-valued continuous functions on $[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\bar{g}(t)\, dt.$$

Let $h \in V$, and define $T : V \to V$ by $T(f) = hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for all $0 \le t \le 1$.

Problem 98. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \operatorname{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2,$$

where the λ_i's are the (not necessarily distinct) eigenvalues of A.

Problem 99. Find an orthogonal matrix whose first row is $(1/3, 2/3, 2/3)$.

Problem 100. Let W be a finite dimensional subspace of an inner product space V. Define $T : V \to V$ by $T(v_1 + v_2) = v_1 - v_2$, where $v_1 \in W$ and $v_2 \in W^\perp$. Prove that T is a self-adjoint unitary operator.